

# Extreme value statistics of eigenvalues of Gaussian random matrices

David S. Dean<sup>1</sup> and Satya N. Majumdar<sup>2</sup>

<sup>1</sup>Laboratoire de Physique Théorique (UMR 5152 du CNRS), Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex 4, France

<sup>2</sup>Laboratoire de Physique Théorique et Modèles Statistiques (UMR 8626 du CNRS), Université Paris-Sud, Bât. 100, 91405 Orsay Cedex, France

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We compute exact asymptotic results for the probability of the occurrence of large deviations of the largest (smallest) eigenvalue of random matrices belonging to the Gaussian orthogonal, unitary, and symplectic ensembles. In particular, we show that the probability that all the eigenvalues of an  $(N \times N)$  random matrix are positive (negative) decreases for large  $N$  as  $\sim \exp[-\beta\theta(0)N^2]$  where the Dyson index  $\beta$  characterizes the ensemble and the exponent  $\theta(0) = (\ln 3)/4 = 0.274653\dots$  is universal. We compute the probability that the eigenvalues lie in the interval  $[\zeta_1, \zeta_2]$  which allows us to calculate the joint probability distribution of the minimum and the maximum eigenvalue. As a by-product, we also obtain exactly the average density of states in Gaussian ensembles whose eigenvalues are restricted to lie in the interval  $[\zeta_1, \zeta_2]$ , thus generalizing the celebrated Wigner semi-circle law to these restricted ensembles. It is found that the density of states generically exhibits an inverse square-root singularity at the location of the barriers. These results are confirmed by numerical simulations. Some of the results presented in detail here were announced in a previous paper [D. S. Dean and S. N. Majumdar, Phys. Rev. Lett. 97, 160201 (2006)].

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## I. INTRODUCTION

Studies of the statistics of the eigenvalues of random matrices have a long history going back to the seminal work of Wigner [1]. Random matrix theory has been successfully applied in various branches of physics and mathematics, including in subjects ranging from nuclear physics, quantum chaos, disordered systems, string theory, and even in number theory [2]. Of particular importance are Gaussian random matrices whose entries are independent Gaussian variables [2]. Depending on the physical symmetries of the problem, three classes of matrices with Gaussian entries arise [2]:  $(N \times N)$  real symmetric [Gaussian orthogonal ensemble (GOE)],  $(N \times N)$  complex Hermitian [Gaussian unitary ensemble (GUE)] and  $(2N \times 2N)$  self-dual Hermitian matrices [Gaussian symplectic ensemble (GSE)]. In these models the probability distribution for a matrix  $M$  in the ensemble is given by

$$p(M) \propto \exp\left(-\frac{\beta}{2}(M, M)\right), \quad (1)$$

where  $(M, M)$  is the inner product on the space of matrices invariant, under orthogonal, unitary, and symplectic transformations, respectively, and the parameter  $\beta$  is the Dyson index. In these three cases the inner products and the Dyson indices are given by

$$(M, M) = \text{Tr}(M^2), \quad \beta = 1 \quad \text{GOE}, \quad (2)$$

$$(M, M) = \text{Tr}(M^*M), \quad \beta = 2 \quad \text{GUE}, \quad (3)$$

$$(M, M) = \text{Tr}(M^\dagger M), \quad \beta = 4 \quad \text{GSE}, \quad (4)$$

where an asterisk denotes the Hermitian conjugate of complex valued matrices and a dagger denotes the symplectic conjugate on quaternion valued matrices. The above qua-

dratic actions are the simplest forms (corresponding to free fields) of matrix models which have been extensively studied in the context of particle physics and field theory.

A central result in the theory of random matrices is the celebrated Wigner semicircle law. It states that for large  $N$  and on average, the  $N$  eigenvalues lie within a finite interval  $[-\sqrt{2N}, \sqrt{2N}]$ , often referred to as the Wigner “sea.” Within this sea, the average density of states has a semicircular form (see Fig. 1) that vanishes at the two edges  $-\sqrt{2N}$  and  $\sqrt{2N}$

$$\rho_{\text{sc}}(\lambda, N) = \sqrt{\frac{2}{N\pi^2} \left[1 - \frac{\lambda^2}{2N}\right]^{1/2}}. \quad (5)$$

The above result means that, if one looks at the density of states of a typical system described by one of the three ensembles above, for a large enough system, it will resemble closely the Wigner semicircle law.

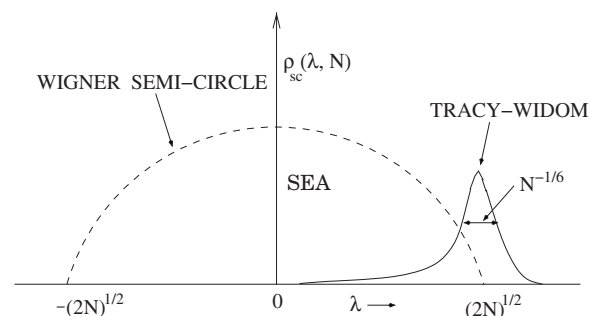


FIG. 1. The dashed line shows the Wigner semicircular form of the average density of states. The largest eigenvalue is centered around its mean  $\sqrt{2N}$  and fluctuates over a scale of width  $N^{-1/6}$ . The probability of fluctuations on this scale is described by the Tracy-Widom distribution (shown schematically).

While the semicircle law provides a global information about how the eigenvalues are typically distributed, unfortunately it does not contain enough information about the probabilities of rare events. The questions concerning rare events have recently come up in different contexts. For example, string theorists have recently been confronted by the possibility that there may be a huge number of effective theories describing our universe. The landscape made up of these theories is called the string landscape. This seemingly embarrassing situation may, however, help to explain certain fine tuning puzzles in particle physics. The basic argument is as follows. Our universe is one which supports intelligent life and this requires that the ratios of certain fundamental constants lie in specific ranges. The fine tuning we observe is thus a necessary condition that we are there to describe it. Other possible universes would have different vacua but there would be no intelligent life to study them. This approach to string theory is called the anthropic principle based string theory and is a subject of current and intense debate. In Refs. [3,4] the authors carried out an analysis of the string landscape based solely on the basis that it is described by a large  $N$  multicomponent scalar potential. Of particular interest is the determination of the typical properties of the vacua in the string landscape based only on assumptions about the dimensionality of the landscape and other simple general features. The motivation for these studies is to determine to what extent the string landscape is determined by large  $N$  statistics and what features depend on the actual structure of the underlying string theory.

One of the main questions posed in Ref. [4] is, for a ( $N \times N$ ) Gaussian random matrix, what is the probability  $P_N$  that all its eigenvalues are positive (or negative)? If we denote the eigenvalues by  $\lambda_i$ ,  $1 \leq i \leq N$ , then the quantity  $P_N$  is simply

$$P_N = \text{Prob}[\lambda_1 \geq 0, \lambda_2 \geq 0, \dots, \lambda_N \geq 0]. \quad (6)$$

From the semicircle law, we know that on an average half the eigenvalues are positive and half of them are negative. Thus the event that all eigenvalues are negative (or positive) is clearly an atypical rare event. This question arises in the so called ‘‘counting problem’’ of local minima in a random multifield potential or a landscape. Given a stationary point of the landscape, if all eigenvalues of the Hessian matrix of the potential are positive, clearly the stationary point is a local minimum. Thus, the probability that all eigenvalues of a random Hessian matrix are positive provides an estimate for the fraction of local minima among the stationary points of the landscape. In particular, the authors of Ref. [4] studied the case where the Hessian matrix was drawn from a GOE ensemble ( $\beta=1$ ). Thus, in this context  $P_N$  is just the probability that a random GOE matrix is positive definite. This probability has also been studied in the mathematics literature [5] and one can easily compute  $P_N$  for smaller values of  $N=1, 2, 3$ . For example, one can show that [5]

$$P_1 = 1/2, \quad P_2 = \frac{2 - \sqrt{2}}{4}, \quad P_3 = \frac{\pi - 2\sqrt{2}}{4\pi}. \quad (7)$$

The interesting question is how  $P_N$  behaves for large  $N$ ? It was argued in Ref. [4] that for large  $N$ ,  $P_N$  decays as  $P_N$

$\sim \exp[-\theta(0)N^2]$ , where the decay constant  $\theta(0)$  was estimated to be  $\approx 1/4$  numerically and via a heuristic argument. The scaling of this probability with  $N^2$  is not surprising and has been alluded to in the literature for a number of years, notably in relation to studies of the distribution of the index (the number of negative eigenvalues) of Gaussian matrices [6,7]. However, an exact expression for  $\theta(0)$  was not available until only recently, when the short form of this paper [8] was published. In Ref. [8] we had shown that for all the three Gaussian ensembles, to leading order in large  $N$

$$P_N \sim \exp[-\beta\theta(0)N^2], \quad \text{where } \theta(0) = \frac{\ln 3}{4} = (0.274653\dots). \quad (8)$$

Interestingly, the probability  $P_N$  has also recently shown up in a rather different problem in mathematics. Dedieu and Maljovich has shown [5] recently that  $P_N$  is exactly equal to the expected number of minima of a random polynomial of degree at most 2 and  $N$  variables. Our result in Ref. [8] thus provided an exact answer to this problem for large  $N$  [5].

To put our results in a more general context, we note that the semicircle law tells us that the average of the maximum (minimum) eigenvalue is  $\sqrt{2N}$  ( $-\sqrt{2N}$ ). However, for finite but large  $N$ , the maximum eigenvalue fluctuates, around its mean  $\sqrt{2N}$ , from one sample to another. Relatively recently Tracy and Widom [9] proved that these fluctuations typically occur over a narrow scale of  $\sim O(N^{-1/6})$  around the upper edge  $\sqrt{2N}$  of the Wigner sea (see Fig. 1). More precisely, they showed [9] that asymptotically for large  $N$ , the scaling variable  $\xi = \sqrt{2N}^{1/6}[\lambda_{\max} - \sqrt{2N}]$  has a limiting  $N$ -independent probability distribution  $\text{Prob}[\xi \leq x] = F_\beta(x)$  whose form depends on the value of the parameter  $\beta=1, 2$ , and 4 characterizing, respectively, the GOE, GUE, and GSE. The function  $F_\beta(x)$ , computed as a solution of a nonlinear differential equation [9], approaches 1 as  $x \rightarrow \infty$  and decays rapidly to zero as  $x \rightarrow -\infty$ . For example, for  $\beta=2$ ,  $F_2(x)$  has the following tails [9]:

$$\begin{aligned} F_2(x) &\rightarrow 1 - O(\exp[-4x^{3/2}/3]) \text{ as } x \rightarrow \infty \\ &\rightarrow \exp[-|x|^3/12] \text{ as } x \rightarrow -\infty. \end{aligned} \quad (9)$$

The probability density function  $dF_\beta/dx$  thus has highly asymmetric tails. The distribution of the minimum eigenvalue simply follows from the fact that  $\text{Prob}[\lambda_{\min} \geq \zeta] = \text{Prob}[\lambda_{\max} \leq -\zeta]$ . Amazingly, the Tracy-Widom distribution has since emerged in a number of seemingly unrelated problems [10] such as the longest increasing subsequence problem [11], directed polymers in (1+1) dimensions [12], various (1+1)-dimensional growth models [13], a class of sequence alignment problems [14], mesoscopic fluctuations in dity metal grains and semiconductor quantum dots [15], and also in finance [16].

The Tracy-Widom distribution describes the probability of typical and small fluctuations of  $\lambda_{\max}$  over a very narrow region of width  $\sim O(N^{-1/6})$  around the mean  $\langle \lambda_{\max} \rangle \approx \sqrt{2N}$ . A natural question is how to describe the probability of atypical and large fluctuations of  $\lambda_{\max}$  around its mean, say over a wider region of width  $\sim O(N^{1/2})$ ? For example, the probability  $P_N$  that all eigenvalues are negative (or positive) is the

same as the probability that  $\lambda_{\max} \leq 0$  (or  $\lambda_{\min} \geq 0$ ). Since  $\langle \lambda_{\max} \rangle \approx \sqrt{2N}$ , this requires the computation of the probability of an extremely rare event characterizing a large deviation of  $\sim -O(N^{1/2})$  to the left of the mean. In Ref. [8] we calculated the exact large deviation function associated with large fluctuations of  $\sim -O(N^{1/2})$  of  $\lambda_{\max}$  to the left of its mean value  $\sqrt{2N}$ . It was shown that for large  $N$  and for all ensembles

$$\text{Prob}[\lambda_{\max} \leq t, N] \sim \exp\left[-\beta N^2 \Phi\left(\frac{\sqrt{2N}-t}{\sqrt{N}}\right)\right], \quad (10)$$

where  $t \sim O(N^{1/2}) \leq \sqrt{2N}$  is located deep inside the Wigner sea. The large deviation function  $\Phi(y)$  is zero for  $y \leq 0$ , but is nontrivial for  $y > 0$  which was computed exactly in Ref. [8]. For small deviations to the left of the mean, taking the  $y \rightarrow 0$  limit of  $\Phi(y)$ , one recovers the left tail of the Tracy-Widom distribution as in Eq. (9). Thus our result for large deviations of  $\sim -O(N^{1/2})$  to the left of the mean is complementary to the Tracy-Widom result for small fluctuations of  $\sim -O(N^{-1/6})$  and the two solutions match smoothly. Also, the probability  $P_N$  that all eigenvalues are negative (or positive) simply follows from the general result in Eq. (10) by putting  $t=0$ ,

$$P_N = \text{Prob}[\lambda_{\max} \leq 0, N] \sim \exp[-\beta \Phi(\sqrt{2})N^2] \quad (11)$$

thus identifying  $\theta(0) = \Phi(\sqrt{2}) = (\ln 3)/4$ , a special case of the general large deviation function.

The purpose of this paper is to provide a detailed derivation of the above results announced in Ref. [8], as well as numerical results in support of our analytical formulas. In addition, we also derive asymptotic results for the joint probability distribution of the minimum and the maximum eigenvalue.

Statistical analysis motivated by anthropic considerations in string theory or random polynomials in mathematics may seem a long way from laboratory based physics. However, similar questions appear naturally also in providing criteria of physical stability in dynamical systems or ecosystems [17,18]. Near a fixed point of a dynamical system, one can linearize the equations of motion and the eigenvalues of the corresponding matrix associated with the linear equations provide important informations about the stability of the fixed point. For example, if all the eigenvalues are negative the fixed point is a stable one. In this context, another important question arises naturally. Suppose that the dynamical system is close to a stable fixed point, i.e., all the eigenvalues are negative. Given this fact, one may further want to know how these negative eigenvalues are distributed. In other words, what is the average density of states of the negative eigenvalues given the fact that one is close to a stable fixed point. In this paper we will calculate the density of states in this conditioned ensemble and we will see that it is quite different from the Wigner semicircle law.

Recently the problem of determining the stability of the critical points of Gaussian random fields in large-dimensional spaces was analyzed [19,20]. The Hessian matrix in this case does not have the statistics of a Gaussian ensemble and the probability that a randomly chosen critical

point is a minimum case be shown to decay as  $\exp(-N\psi)$  and the exponent  $\psi$  can be explicitly calculated in terms of the two point correlation function of the Gaussian field. Thus the scaling in  $N$  is quite different to the random matrix case and this scaling obviously makes minima much more likely and yields a more usual thermodynamic scaling of the entropy of critical points [19]. The statistics of the Gaussian field problem are perhaps more relevant to statistical landscape scenarios in string theory.

The paper is organized as follows. In Sec. II we begin by recalling the Coulomb gas representation of the distribution of eigenvalues of Gaussian matrices and show how our problem can be formulated by placing a hard wall constraint on the Coulomb gas. This is the key step in the method and the technique has since been applied to analyze the statistics of critical points of Gaussian random fields [19] and also to study the probability of rare fluctuations of the maximal eigenvalue of Wishart random matrices [21]. We then show how in the large  $N$  limit the problem can be solved using a saddle-point computation of a functional integral and we discuss the features of our analytic results. In Sec. III, we extend this method to compute the asymptotic joint probability distribution of the minimum and the maximum eigenvalue. This requires studying the Coulomb gas confined between two hard walls. In Sec. IV we carry out some numerical work to confirm our predictions about the probability of extreme deviations of the maximal eigenvalue. We show how the Coulomb gas formulation can again be exploited even for numerical purposes. Finally we present our conclusions in Sec. V.

## II. THE COULOMB GAS FORMULATION AND THE PROBABILITY OF RARE FLUCTUATIONS

The joint probability density function (PDF) of the eigenvalues of an  $N \times N$  Gaussian matrix is given by the classic result of Wigner [1,2]

$$P(\lambda_1, \lambda_2, \dots, \lambda_N) = B_N \exp\left[-\frac{\beta}{2} \left( \sum_{i=1}^N \lambda_i^2 - \sum_{i \neq j} \ln(|\lambda_i - \lambda_j|) \right)\right], \quad (12)$$

where  $B_N$  normalizes the pdf and  $\beta=1, 2$ , and  $4$  correspond, respectively, to the GOE, GUE, and GSE. The joint law in Eq. (12) allows one to interpret the eigenvalues as the positions of charged particles, repelling each other via a 2D Coulomb potential (logarithmic); they are confined on a 1D line and each is subject to an external harmonic potential. The parameter  $\beta$  that characterizes the type of ensemble can then be interpreted as the inverse temperature.

Once the joint PDF is known explicitly, other statistical properties of a random matrix can, in principle, be derived from this joint PDF. In practice, however, this is often a technically daunting task. For example, suppose we want to compute the average density of states of the eigenvalues defined as  $\rho(\lambda, N) = \sum_{i=1}^N \langle \delta(\lambda - \lambda_i) \rangle / N$ , which counts the average number of eigenvalues between  $\lambda$  and  $\lambda + d\lambda$  per unit length. The angled bracket  $\langle \dots \rangle$  denotes an average over the joint PDF. It then follows that  $\rho(\lambda, N)$  is simply the marginal of

the joint PDF, i.e., we fix one of the eigenvalues (say the first one) at  $\lambda$  and integrate the joint PDF over the rest of the  $(N-1)$  variables.

$$\rho(\lambda, N) = \frac{1}{N} \sum_{i=1}^N \langle \delta(\lambda - \lambda_i) \rangle = \int_{-\infty}^{\infty} \prod_{i=2}^N d\lambda_i P(\lambda, \lambda_2, \dots, \lambda_N). \quad (13)$$

Wigner computed this marginal and showed [1] that for large  $N$  and for all  $\beta$  it has the semicircular form in Eq. (5).

Here we are interested in calculating the probability  $Q_N(\zeta)$  that all eigenvalues are greater than some value  $\zeta$ . This probability is clearly also equal to the cumulative probability that the minimum eigenvalue  $\lambda_{\min} = \min(\lambda_1, \lambda_2, \dots, \lambda_N)$  is greater than  $\zeta$ , i.e.,

$$\begin{aligned} Q_N(\zeta) &= \text{Prob}[\lambda_1 \geq \zeta, \lambda_2 \geq \zeta, \dots, \lambda_N \geq \zeta] \\ &= \text{Prob}[\lambda_{\min} \geq \zeta, N]. \end{aligned} \quad (14)$$

Since the Gaussian random matrix has the  $x \rightarrow -x$  symmetry, it follows that the maximum eigenvalue  $\lambda_{\max}$  has the same statistics as  $-\lambda_{\min}$ . Hence, it follows that

$$\text{Prob}[\lambda_{\max} \leq t, N] = Q_N(\zeta = -t). \quad (15)$$

Hence, knowing  $Q_N(\zeta)$  will also allow us to compute the cumulative distribution of the maximum. Also, note that the probability  $P_N$  that all eigenvalues are positive (or negative), as defined in the introduction, is simply

$$P_N = Q_N(0). \quad (16)$$

In what follows, we will compute  $Q_N(\zeta)$  in the scaling limit where  $\zeta \sim \sqrt{N}$  for large  $N$ . For this we will employ the saddle-point method in the framework of the Coulomb gas.

By definition

$$Q_N(\zeta) = \int_{\zeta}^{\infty} \cdots \int_{\zeta}^{\infty} P(\lambda_1, \lambda_2, \dots, \lambda_N) d\lambda_1 d\lambda_2 \cdots d\lambda_N, \quad (17)$$

where  $P$  is the joint PDF in Eq. (12). This multiple integral can be written as a ratio

$$Q_N(\zeta) = \frac{Z_N(\zeta)}{Z_N(-\infty)}, \quad (18)$$

where the partition function  $Z_N(\zeta)$  is defined as

$$\begin{aligned} Z_N(\zeta) &= \int_{\zeta}^{\infty} \cdots \int_{\zeta}^{\infty} \exp \left[ -\frac{\beta}{2} \left( \sum_{i=1}^N \lambda_i^2 - \sum_{i \neq j} \ln(|\lambda_i - \lambda_j|) \right) \right] \\ &\quad \times d\lambda_1 d\lambda_2 \cdots d\lambda_N. \end{aligned} \quad (19)$$

Note that the normalization constant  $B_N = 1/Z_N(-\infty)$ . Clearly, for any finite  $\zeta$ ,  $Z_N(\zeta)$  represents the partition function of the Coulomb gas which is constrained in the region  $[\zeta, \infty]$ , i.e., it has a hard wall at  $\zeta$  ensuring that there are no eigenvalues to the left of  $\zeta$ .

From the semicircle law, it is evident a typical eigenvalue scales as  $\sqrt{N}$  for large  $N$ . This is also evident from Eq. (19) where the Coulomb interaction term typically scales as  $N^2$

for large  $N$  while the energy corresponding to the external potential scales as  $\lambda^2 N$ . In order that they balance, it follows that typically  $\lambda \sim \sqrt{N}$ . It is therefore natural to rescale  $\mu_i = \lambda_i / \sqrt{N}$  so that the rescaled eigenvalue  $\mu_i \sim O(1)$  for large  $N$ . In terms of the rescaled eigenvalues, the partition function reads

$$Z_N(\zeta) \propto \int_{\mu_i > (\zeta/\sqrt{N})}^{\infty} \prod_{i=1}^N d\mu_i \exp[-\beta H(\mu)],$$

where the Hamiltonian of the Coulomb gas is

$$H(\mu) = \frac{N}{2} \sum_{i=1}^N \mu_i^2 - \frac{1}{2} \sum_{i \neq j} \ln(|\mu_i - \mu_j|). \quad (20)$$

### A. Functional integral, large $N$ saddle-point analysis and the constrained charge density

We first define the normalized (to unity) spatial density field of the particles  $\mu_i$  as

$$\rho(\mu) = \frac{1}{N} \sum_{i=1}^N \delta(\mu - \mu_i). \quad (21)$$

This is just a ‘‘counting’’ function so that  $\rho(\mu) d\mu$  counts the fraction of eigenvalues between  $\mu$  and  $\mu + d\mu$ . The energy of a configuration of the  $\mu_i$ 's can be expressed in terms of the density  $\rho$  as

$$H[\rho] = N^2 \mathcal{E}[\rho], \quad (22)$$

where

$$\begin{aligned} \mathcal{E}[\rho] &= \frac{1}{2} \int d\mu \mu^2 \rho(\mu) - \frac{1}{2} \int d\mu d\mu' \rho(\mu) \rho(\mu') \ln(|\mu - \mu'|) \\ &\quad + \frac{1}{2N} \int d\mu \rho(\mu) \ln[l(\mu)]. \end{aligned} \quad (23)$$

The last term above removes the self interaction energy from the penultimate term and  $l(\mu)$  represents a position-dependent cutoff. Dyson [23] argued that  $l(\mu) \sim 1/\rho(\mu)$  and hence this correction term has the form

$$\int d\mu \rho(\mu) \ln[l(\mu)] = - \int d\mu \rho(\mu) \ln[\rho(\mu)] + C', \quad (24)$$

where  $C'$  is a constant that cannot be determined by this argument, but one may assume that it is independent of the position of the hard wall. However, in this paper, we will be interested only in the leading  $O(N^2)$  behavior and hence precise value of the constant  $C'$  does not matter.

The hard wall constraint can now be implemented simply by the condition

$$\rho(\mu) = 0 \quad \text{for } \mu < \frac{\zeta}{\sqrt{N}}. \quad (25)$$

The partition function may now be written as a functional integral over the density field  $\rho$  as

$$Z_N(\zeta) = \int d[\rho] J[\rho] \exp(-\beta N^2 \mathcal{E}[\rho]), \quad (26)$$

where  $J[\rho]$  is the Jacobian involved in changing from the coordinates  $\mu_i$  to the density field  $\rho$ . Physically this Jacobian takes into account the entropy associated with the density field  $\rho$ . For the sake of completeness we will rederive a familiar form of the Jacobian  $J[\rho]$ . Clearly  $J$  can be written, up to a constant prefactor  $D_N$ , as

$$J[\rho] = D_N \int \prod_{i=1} d\mu_i \delta \left[ N\rho(\mu) - \sum_i \delta(\mu - \mu_i) \right], \quad (27)$$

where the integration range of the  $\mu_i$  above are restricted to the appropriate region. One now proceeds by making a functional Fourier transform representation of the  $\delta$  function at each point  $\mu$ . This gives

$$J[\rho] = D'_N \int \prod_{i=1} d\mu_i d[g] \exp \left\{ \int d\mu g(\mu) \left[ N\rho(\mu) - \sum_i \delta(\mu - \mu_i) \right] \right\}, \quad (28)$$

where each  $g(\mu)$  integral is along the imaginary axis and  $D'_N$  is a constant prefactor. The integral over the  $\mu_i$  may now be carried out giving

$$\begin{aligned} J[\rho] &= D'_N \int d[g] \exp \left[ N \int d\mu g(\mu) \rho(\mu) \right] \prod_{i=1}^N d\mu_i \exp[-g(\mu_i)] \\ &= D'_N \int d[g] \exp \left( N \int d\mu g(\mu) \rho(\mu) + N \ln \left\{ \int d\mu \exp[-g(\mu)] \right\} \right). \end{aligned} \quad (29)$$

The above functional integral over  $g$  can be evaluated by saddle point for large  $N$  and the corresponding saddle-point equation (obtained via stationarity with respect to  $g$ ) is

$$\rho(\mu) = \frac{\exp[-g(\mu)]}{\int d\mu' \exp[-g(\mu')]}. \quad (30)$$

We see that the normalization

$$\int d\mu \rho(\mu) = 1 \quad (31)$$

is respected. Substituting in this solution for  $g$  we find that

$$J[\rho] = D'_N \exp \left\{ -N \int d\mu \rho(\mu) \ln[\rho(\mu)] \right\}, \quad (32)$$

from which we can read off the entropy corresponding to the density field  $\rho$ .

Putting this all together we find

$$\begin{aligned} Z_N(\zeta) &= A_N \int d[\rho] \delta \left[ \int d\mu \rho(\mu) - 1 \right] \\ &\times \exp \left\{ -N \int d\mu \rho(\mu) \ln[\rho(\mu)] - \beta N^2 \mathcal{E}[\rho] \right\}, \end{aligned} \quad (33)$$

where  $A_N$  is a prefactor and the delta function enforces the normalization condition in Eq. (31). One simple way to incorporate this delta function constraint is to introduce a Lagrange multiplier  $C$  (corresponding to writing the delta function constraint in the Fourier representation) and rewrite the partition function as

$$\begin{aligned} Z_N(\zeta) &= A'_N \int dC d[\rho] \exp \left[ -\beta N^2 \Sigma[\rho] - N \left( 1 - \frac{\beta}{2} \right) \right. \\ &\left. \times \int d\mu \rho(\mu) \ln[\rho(\mu)] \right], \end{aligned} \quad (34)$$

where  $A'_N$  is a constant and the “renormalized” action is explicitly

$$\begin{aligned} \Sigma[\rho] &= \frac{1}{2} \int d\mu \mu^2 \rho(\mu) - \frac{1}{2} \int d\mu d\mu' \rho(\mu) \rho(\mu') \ln(|\mu - \mu'|) \\ &+ C \left[ \int d\mu \rho(\mu) - 1 \right]. \end{aligned} \quad (35)$$

The  $O(N)$  term in Eq. (34) includes both the entropy of the density field and the self-energy subtraction following the Dyson prescription discussed before. While this prescription is difficult to justify rigorously, we need not pursue this issue here since we are interested only in the leading  $O(N^2)$  behavior.

The functional integral in Eq. (34) can thus be evaluated by the saddle-point method in the variable  $N^2$  and the term of order  $N$  coming from the entropy and self-energy subtraction is negligible in the large  $N$  limit. We thus find that in the saddle-point analysis, for large  $N$ ,

$$Z_N(\zeta) = \exp[-\beta N^2 \Sigma[\rho_c] + O(N)], \quad (36)$$

where  $\rho_c(\mu)$  is the density field that minimizes the action  $\Sigma[\rho]$  in Eq. (35). Minimizing  $\Sigma[\rho]$ , it follows that  $\rho_c(\mu)$  satisfies the integral equation

$$\frac{\mu^2}{2} + C = \int_z^\infty d\mu' \rho_c(\mu') \ln(|\mu - \mu'|), \quad (37)$$

where we have introduced the scaled variable  $z = \zeta / \sqrt{N}$ . The normalization and boundary conditions for  $\rho_c(\mu)$ , in terms of the scaled variable  $z$ , are

$$\int_z^\infty d\mu \rho_c(\mu) = 1, \quad \rho_c(\mu) = 0 \text{ for } \mu < z. \quad (38)$$

The partition function then reads

$$Z_N(z) = \exp[-\beta N^2 S(z) + O(N)], \quad (39)$$

where

$$S(z) = \min_{\rho} \{ \Sigma[\rho] \} = \Sigma[\rho_c]. \quad (40)$$

Let us remark here that in the unconstrained case ( $z = -\infty$ ) the Wigner semicircle law can also be obtained from the saddle-point method [2]. The Coulomb gas and the saddle-point method was also used for a class of constrained GUE (fixed trace ensemble) problems [22]. However, to our knowledge the constrained problem in presence of a hard wall has never been analyzed using this approach. In addition, the effective free energy is not extensive: it scales as  $N^2$  rather than  $N$  due to the long-range nature of the logarithmic interparticle interaction. The entropy term is extensive and is contained in the  $O(N)$  term in Eq. (39). However, the entropy is subdominant, and the free energy is dominated by the energetic component.

Differentiating Eq. (37) with respect to  $\mu$  we get

$$\mu = \mathcal{P} \int_z^{\infty} d\mu' \frac{\rho_c(\mu')}{\mu - \mu'}, \quad (41)$$

where  $\mathcal{P}$  indicates the Cauchy principal part. It is convenient to shift the variable  $\mu$  by writing

$$\mu = x + z \quad (42)$$

and where  $x \geq 0$  is now positive and  $x=0$  denotes the location of the infinite barrier. In terms of the shifted variable, we denote the density field as

$$\rho_c(\mu = x + z) = f(x; z). \quad (43)$$

Equation (41) then becomes

$$x + z = \mathcal{P} \int_0^{\infty} dx' \frac{f(x'; z)}{x - x'}. \quad (44)$$

This integral equation is of the general form

$$g(x) = (H_+ f)(x) = \mathcal{P} \int_0^{\infty} dx' \frac{f(x')}{x - x'}, \quad (45)$$

with  $g(x) = x + z$ . The right-hand side (RHS) of the above equation is just the semi-infinite Hilbert transform of the function  $f(x)$ . The main technical challenge is to invert this half Hilbert transform. Fortunately this inversion can be done for arbitrary  $g(x)$  using Tricomi's theorem [24]. To apply Tricomi's theorem, we first assume, to be verified a posteriori, that  $f(x)$  has a finite support over  $[0, L]$ , so that the integral in Eq. (44) can be cutoff at  $L$  at the upper edge. Then the solution for  $f(x)$  is given by Tricomi's theorem [24]

$$f(x) = - \frac{1}{\pi^2 \sqrt{x(L-x)}} \left\{ \mathcal{P} \int_0^L dx' \sqrt{x'(L-x')} \frac{g(x')}{x-x'} + C' \right\}, \quad (46)$$

where  $C'$  is an arbitrary constant which can be fixed by demanding that  $f(x)$  vanishes at  $x=L$ . Applying this formula to our case with  $g(x) = x + z$  and performing the integral in Eq. (46) using MATHEMATICA, we get

$$f(x; z) = \frac{1}{2\pi\sqrt{x}} \sqrt{L(z) - x} [L(z) + 2x + 2z], \quad x \in [0, L(z)],$$

$$f(x; z) = 0, \quad x < 0 \text{ and } x > L(z), \quad (47)$$

where we have made the  $z$  dependence of  $L(z)$  explicit. The value of  $L(z)$  is now determined from the normalization of  $f(x; z)$ , i.e.,  $\int_0^{L(z)} f(x; z) dx = 1$  and one gets

$$L(z) = \frac{2}{3} [\sqrt{z^2 + 6} - z] \quad (48)$$

which is always positive. We also see that in terms of the variable  $\mu = x + z$ , the support of  $\rho_c(\mu)$  is between the barrier location at  $z$  and an upper edge at  $\mu = z + L(z) = \frac{2}{3} [\sqrt{z^2 + 6} + 2z]$ .

Let us make a few observations about the constrained charge density  $\rho_c(\mu = x + z) = f(x; z)$  in Eqs. (47) and (48). Physically, the charge density  $f(x; z)$  must be positive for all  $x$  including  $x=0$ . Now, as  $x \rightarrow 0$ , i.e., as one approaches the infinite barrier from the right, it follows from Eq. (47) that  $f(x; z)$  diverges as  $x^{-1/2}$ , i.e., the charges accumulate near the barrier. However, for it to remain positive at  $x=0$ , it follows from Eq. (47) that we must have  $L(z) + 2z \geq 0$ . This happens, using the expression of  $L(z)$  from Eq. (48), only for  $z \geq -\sqrt{2}$ . Thus the result in Eq. (47) is valid only for  $z \geq -\sqrt{2}$ . To understand the significance of  $z \geq -\sqrt{2}$ , we note that when  $z = -\sqrt{2}$ , i.e., when the barrier is placed exactly at the left-most edge of the Wigner sea, we recover the Wigner semicircle law from Eqs. (47) and (48). We get  $L(z = -\sqrt{2}) = 2\sqrt{2}$  (the support of the full semicircle), i.e.,  $f(x; -\sqrt{2})$  is nonzero for  $0 \leq x \leq 2\sqrt{2}$ . In terms of the original variable  $\mu = x + z = x - \sqrt{2}$ ,  $\rho_c(\mu)$  is nonzero in the region  $-\sqrt{2} \leq \mu \leq \sqrt{2}$  and is given by, within this sea to the right of the barrier, in terms of the original variable  $\mu$

$$\rho_c(\mu) = \frac{1}{\pi} \sqrt{2 - \mu^2}. \quad (49)$$

When  $z$  becomes smaller than  $-\sqrt{2}$ , the solution in Eq. (47) remains unchanged from its semicircular form at  $z = -\sqrt{2}$ . In other words, for  $z \leq -\sqrt{2}$ , the solution sticks to its form at  $z = -\sqrt{2}$ . Physically, this means that if the barrier is placed to the left of the lower edge of the Wigner sea, it has no effect on the charge distribution. Note that the above fact, in conjunction with Eq. (40), indicates that

$$S(z) = S(-\sqrt{2}) \text{ for all } z \leq -\sqrt{2} \quad (50)$$

including, in particular,

$$S(-\infty) = S(-\sqrt{2}) \quad (51)$$

a result that we will use later.

The charge density  $\rho_c(\mu = x + z) = f(x; z)$  changes its shape in an interesting manner as one changes the barrier location  $z$ . For  $z > -\sqrt{2}$  the global maximum of  $f(x; z)$  is always at  $x=0$  (at the barrier) where it has a  $1/\sqrt{x}$  integrable singularity. For  $-\sqrt{2} < z < -\sqrt{3}/4$ , the density is nonmonotonic and in addition to the square root divergence at  $x=0$ ,  $f(x; z)$  develops a local minimum and a local maximum, respectively, at  $x = (L \mp \sqrt{3L^2 - 8zL})/4$ . For  $z > -\sqrt{3}/4$ , the density  $f(x; z)$  decreases monotonically with increasing  $x$ . In Fig. 2 we show the behavior of  $f(x; z)$  for three representative values of  $z$ . An interesting point about the eigenvalue distribution

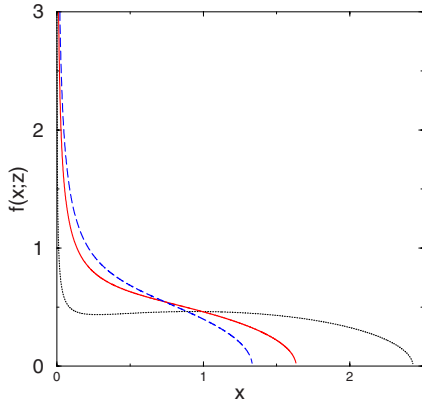


FIG. 2. (Color online): Plots of the density of states  $f(x; z)$  as a function of the shifted variable  $x$  for  $z = -1$  (dotted),  $z = 0$  (solid), and  $z = 0.5$  (dashed).

$f(x; z)$  given Eq. (47) is that, for  $z > -\sqrt{2}$  there is a strong accumulation of eigenvalues at the barrier location  $x = 0$  or, equivalently, at  $\mu = z$ . In the case  $z = 0$  if one thinks of the eigenvalues as being associated about the vacuum of a (stable) field theory then there is an accumulation of modes of mass close to zero, a fact that may have consequences in the context of anthropic principal based string theory or in other physical systems where only stable configurations can be observed.

### B. Large deviations of the maximum or the minimum

Having computed the constrained charged density  $\rho_c(\mu)$ , we next calculate the action  $S(z) = \Sigma[\rho_c]$  at the saddle point. In order to calculate  $\Sigma[\rho_c]$  from Eq. (35), we will use the explicit saddle-point solution  $\rho_c(\mu = x + z) = f(x; z)$  obtained in Eq. (47). In fact, one can simplify the expression of the saddle-point action by using the integral Eq. (37) satisfied by the saddle-point solution. We multiply Eq. (37) by  $\rho_c(\mu)$  and integrate over  $\mu$ . Using the normalization  $\int \rho_c(\mu) d\mu = 1$  one gets

$$\int d\mu d\mu' \rho_c(\mu) \rho_c(\mu') \ln(|\mu - \mu'|) = C + \frac{1}{2} \int \mu^2 \rho_c(\mu) d\mu, \quad (52)$$

where  $C$  is the Lagrange multiplier to be determined. Substituting this result in Eq. (35) and the fact  $\int \rho_c(\mu) d\mu = 1$  we get

$$\Sigma[\rho_c] = -\frac{1}{2}C + \frac{1}{4} \int \mu^2 \rho_c(\mu) d\mu. \quad (53)$$

To determine the Lagrange multiplier  $C$ , we put  $\mu = z$  in Eq. (37). In the integral on the RHS of Eq. (37) we then make the usual shift,  $\mu' = z + x'$  and use  $\rho_c(\mu') = f(x'; z)$  to get

$$C = -\frac{1}{2}z^2 + \int_0^{L(z)} dx' f(x'; z) \ln(x'), \quad (54)$$

where  $f(x; z)$  is explicitly given in Eq. (47) and  $L(z)$  in Eq. (48). Substituting the expression of  $C$  in Eq. (53) and  $\rho_c(\mu') = f(x'; z)$  we finally get

$$\Sigma[\rho_c] = \frac{1}{4}z^2 - \frac{1}{2} \int_0^{L(z)} dx \ln(x) f(x; z) + \frac{1}{4} \int_0^{L(z)} dx (x+z)^2 f(x; z) \quad (55)$$

with  $f(x; z)$  given explicitly in Eq. (47). The integrals can again be performed explicitly using MATHEMATICA and using Eq. (40) we get the following expression for the saddle-point action:

$$S(z) = \frac{1}{216} \{ 72z^2 - 2z^4 + (30z + 2z^3) \sqrt{6+z^2} + 27[3 + \ln(1296) - 4 \ln(-z + \sqrt{6+z^2})] \}. \quad (56)$$

The partition function  $Z_N(\zeta = \sqrt{N}z)$  is then given by Eq. (39). Note also, using Eq. (51), we have

$$Z_N(-\infty) = \exp[-\beta N^2 S(-\sqrt{2}) + O(N)], \quad (57)$$

where  $S(-\sqrt{2}) = [3 + \ln(4)]/8$  from Eq. (56). Taking the ratio in Eq. (18) gives us  $Q_N(\zeta)$ , the probability  $Q_N(\zeta)$  that all eigenvalues are to the right of  $\zeta = \sqrt{N}z$

$$Q_N(\zeta) = \exp \left[ -\beta N^2 \theta \left( \frac{\zeta}{\sqrt{N}} \right) + O(N) \right], \quad (58)$$

where  $\theta(z) = S(z) - S(-\sqrt{2})$  is given by

$$\theta(z) = \frac{1}{108} \{ 36z^2 - z^4 + (15z + z^3) \sqrt{6+z^2} + 27[\ln(18) - 2 \ln(-z + \sqrt{6+z^2})] \}. \quad (59)$$

The probability that all eigenvalues are positive (or negative) is simply

$$P_N = Q_N(\zeta = 0) \approx \exp[-\beta \theta(0) N^2], \quad (60)$$

where

$$\theta(0) = \frac{\ln(3)}{4} = 0.274653 \dots \quad (61)$$

Finally let us turn to the large deviation function associated with large negative fluctuations of  $\sim -O(N^{1/2})$  of  $\lambda_{\max}$  to the left of its mean value  $\sqrt{2N}$ . Substituting the expression for  $Q_N(\zeta)$  from Eq. (58) in Eq. (15) we get

$$\begin{aligned} \text{Prob}[\lambda_{\max} \leq t, N] &= Q_N(\zeta = -t) \\ &= \exp \left[ -\beta N^2 \theta \left( -\frac{t}{\sqrt{N}} \right) + O(N) \right]. \end{aligned} \quad (62)$$

Noting that  $\langle \lambda_{\max} \rangle = \sqrt{2N}$ , it is useful to center the distribution around the mean and rewrite Eq. (62) as

$$\text{Prob}[\lambda_{\max} \leq t, N] = \exp \left[ -\beta N^2 \Phi \left( \frac{\sqrt{2N} - t}{\sqrt{N}} \right) \right], \quad (63)$$

where the large deviation function  $\Phi(y) = \theta(y - \sqrt{2})$  with  $\theta(z)$  given explicitly in Eq. (59).

One can easily work out the asymptotic behavior of  $\Phi(y)$  for small and large  $y$ . For example, it is easy to see that

$$\begin{aligned}\Phi(y) &\approx \frac{y^3}{6\sqrt{2}} \text{ as } y \rightarrow 0, \\ &\approx \frac{y^2}{2} \text{ as } y \rightarrow \infty.\end{aligned}\quad (64)$$

In particular, when  $\sqrt{2N-t} \ll \sqrt{N}$ , i.e., we are rather close to the right edge  $\langle \lambda_{\max} \rangle = \sqrt{2N}$  of the Wigner sea (see Fig. 3), it follows that the scaling variable  $y = (\sqrt{2N-t})/\sqrt{N} \ll 1$ . Hence substituting the small  $y$  behavior of  $\Phi(y) \approx y^3/6\sqrt{2}$  from Eq. (64) in Eq. (63), it follows that in this regime

$$\text{Prob}[\lambda_{\max} \leq t, N] \approx \exp\left\{-\frac{\beta}{24} [|\sqrt{2N}^{1/6}(t - \sqrt{2N})|]^3\right\}.\quad (65)$$

Note that this matches exactly with the left tail behavior of the Tracy-Widom limiting distribution for all the three cases  $\beta=1, 2$ , and  $4$  [25]. For example, for  $\beta=2$ , one can easily verify by comparing Eqs. (65) and (9). This is to be expected because the Tracy-Widom distribution describes the distribution of  $\lambda_{\max}$  around its mean  $\sqrt{2N}$  over a scale  $\sim O(N^{-1/6})$ . If we want to investigate the probability of negative fluctuations of order  $(\sqrt{2N-t}) \gg N^{-1/6}$ , we need to look at the left tail of the Tracy-Widom distribution. On the other hand, those negative fluctuations of order  $N^{-1/6} \ll (\sqrt{2N-t}) \ll \sqrt{N}$  are described by the small argument behavior of the large deviation function. These two behaviors thus should smoothly match. As we verified above, they do indeed match smoothly, thus providing another confirmation of our exact result. Moreover, our large deviation function provides an alternative way to compute the left tail of the Tracy-Widom distribution for all  $\beta$ .

### III. COULOMB GAS BOUNDED BY TWO WALLS: THE JOINT PROBABILITY DISTRIBUTION OF $\lambda_{\min}$ AND $\lambda_{\max}$

In this section we compute, by the Coulomb gas method, the probability  $R_N(\zeta_1, \zeta_2)$  that all eigenvalues are in the interval  $[\zeta_1, \zeta_2]$  where  $\zeta_2 \geq \zeta_1$ . Evidently, in the limit  $\zeta_2 \rightarrow \infty$ ,  $R_N(\zeta, \infty) = Q_N(\zeta)$  which was computed in the previous section. Clearly,  $R_N(\zeta_1, \zeta_2)$  is also the cumulative probability that  $\lambda_{\min} \geq \zeta_1$  and  $\lambda_{\max} \leq \zeta_2$ , i.e.,

$$\begin{aligned}R_N(\zeta_1, \zeta_2) &= \text{Prob}[\zeta_1 \leq \lambda_1 \leq \zeta_2, \zeta_1 \leq \lambda_2 \leq \zeta_2, \dots, \zeta_1 \leq \lambda_N \\ &\leq \zeta_2] = \text{Prob}[\lambda_{\min} \geq \zeta_1, \lambda_{\max} \leq \zeta_2].\end{aligned}\quad (66)$$

In other words,  $R_N(\zeta_1, \zeta_2)$  provides the joint probability distribution of the minimum and the maximum eigenvalue. By definition,

$$R_N(\zeta_1, \zeta_2) = \int_{\zeta_1}^{\zeta_2} \cdots \int_{\zeta_1}^{\zeta_2} P(\lambda_1, \lambda_2, \dots, \lambda_N) d\lambda_1 d\lambda_2 \cdots d\lambda_N,\quad (67)$$

where  $P$  is the joint PDF in Eq. (12). As in the previous section, this multiple integral can be written as a ratio of two partition functions

$$R_N(\zeta_1, \zeta_2) = \frac{\Omega_N(\zeta_1, \zeta_2)}{Z_N(-\infty)},\quad (68)$$

where

$$\begin{aligned}\Omega_N(\zeta_1, \zeta_2) &= \int_{\zeta_1}^{\zeta_2} \cdots \int_{\zeta_1}^{\zeta_2} \exp\left[-\frac{\beta}{2} \left(\sum_{i=1}^N \lambda_i^2 - \sum_{i \neq j} \ln(|\lambda_i - \lambda_j|)\right)\right] \\ &\times d\lambda_1 d\lambda_2 \cdots d\lambda_N\end{aligned}\quad (69)$$

and  $Z_N(-\infty) = \Omega_N(-\infty, \infty)$  is the same normalization constant as in Eq. (18). Thus,  $\Omega_N(\zeta_1, \zeta_2)$  represents the partition function of the Coulomb gas that is sandwiched in the region  $[\zeta_1, \zeta_2]$  bounded by the two hard walls at its boundaries.

We then evaluate this partition function in the large  $N$  limit using the saddle-point method. The formalism is exactly same as in the previous section. We first define a counting function  $\rho(\mu)$  that is nonzero only in the region  $\frac{\zeta_1}{\sqrt{N}} \leq \mu \leq \frac{\zeta_2}{\sqrt{N}}$  and is zero outside. The rest of the calculation is similar as in the previous section, except that all the integrals run over the region  $\mu \in [z_1, z_2]$ , where  $z_1 = \zeta_1/\sqrt{N}$  and  $z_2 = \zeta_2/\sqrt{N}$ . The action  $\Sigma[\rho]$  is exactly as in Eq. (35). Thus the partition function, in the large  $N$  limit, behaves as

$$\Omega_N(\zeta_1, \zeta_2) = \exp\{-\beta N^2 \Sigma[\rho_c] + O(N)\},\quad (70)$$

where the saddle-point density  $\rho_c(\mu)$ , in terms of scaled variables  $z_1 = \zeta_1/\sqrt{N}$  and  $z_2 = \zeta_2/\sqrt{N}$ , satisfies the integral equation

$$\mu = \mathcal{P} \int_{z_1}^{z_2} d\mu' \frac{\rho_c(\mu')}{\mu - \mu'},\quad (71)$$

where  $\mathcal{P}$  indicates the Cauchy principle part. Next we introduce the shift

$$\mu = z_1 + x\quad (72)$$

and define  $W = z_2 - z_1$ . Since  $z_1 \leq \mu \leq z_2$ , it follows that  $0 \leq x \leq W$ . Thus  $x=0$  denotes the location of the left barrier and  $x=W$  denotes the location of the right barrier. In terms of the shifted variable, we rewrite the density field as

$$\rho_c(\mu = z_1 + x) = f(x; z_1, W).\quad (73)$$

Equation (71) then reduces to the integral equation

$$x + z_1 = \mathcal{P} \int_0^W dx' \frac{f(x'; z_1, W)}{x - x'}.\quad (74)$$

The integral Eq. (74) can again be solved using Tricomi's theorem. Note that this equation has almost similar form as Eq. (44) except that the integral on the RHS of Eq. (74) runs up to  $W$ . From the solution of Eq. (44) presented in Eq. (47) we learned that the density  $f$  is nonzero only for  $0 \leq x \leq L(z)$  and is zero for  $x > L(z)$ , where  $L(z) = \frac{2}{3}[\sqrt{z^2 + 6} - z]$ . So, comparing to Eq. (74) we see that there are two possibilities.

(i) If  $W = z_2 - z_1 > L(z_1)$ , the solution  $f(x; z_1, W)$  will be exactly the same as  $f(x; z_1)$  presented in Eq. (47). In this case, the Coulomb gas does not feel the presence of the right barrier at  $z_2$ . In other words, the solution  $f(x; z_1, W) = f(x; z_1, \infty)$  is completely independent of  $W$  and one can



effectively put  $W \rightarrow \infty$ , i.e., put the right barrier at infinity. Thus in the case, the charge density diverges as  $x^{-1/2}$  at the left barrier and vanishes at  $x=L(z_1)$ .

(ii) If  $W=z_2-z_1 < L(z_1)$ , then the solution will be given by Eq. (46) with  $L$  replaced by  $W$ . Using  $g(x)=x+z_1$  in Eq. (46) and performing the integral on the RHS we get

$$f(x; z_1, W) = \frac{1}{8\pi\sqrt{x(W-x)}} [W^2 + 4W(x+z_1) - 8x(x+z_1) + B'], \quad (75)$$

where  $B'$  is an arbitrary constant. The normalization condition  $\int_0^W f(x; z_1, W) dx = 1$  fixes the constant  $B' = 8$ . In this case, the charge density diverges (with a square root singularity) at the locations of both the left barrier ( $x=0$ ) and the right barrier ( $x=W$ ).

Thus, putting (i) and (ii) together, we find that the solution for the equilibrium charge density  $f(x; z_1, W)$  for the Coulomb gas sandwiched between two barriers is given by

$$f(x; z_1, W) = \frac{1}{8\pi\sqrt{x(l-x)}} [l^2 + 4l(x+z_1) - 8x(x+z_1) + 8] \quad \text{for } 0 \leq x \leq l, \quad (76)$$

where

$$l = \min \left[ W = z_2 - z_1, L(z_1) = \frac{2}{3}(\sqrt{z_1^2 + 6} - z_1) \right]. \quad (77)$$

Clearly, in the limit  $W \rightarrow \infty$ , we indeed recover the results of the previous section. Summarizing (see Fig. 3), if one fixes the left barrier at  $z_1$  and varies the position of the right barrier  $z_2$  (equivalently by varying the distance  $W=z_2-z_1$  between the two walls), one finds that the charge density at the left barrier always diverges. On the other hand, the behavior of the density near the right barrier undergoes a sudden change as  $W$  increases beyond a critical value  $W_c = L(z_1) = \frac{2}{3}(\sqrt{z_1^2 + 6} - z_1)$ . The density at the right wall diverges as long as  $W < W_c$ , i.e.,  $z_2 < z_1 + L(z_1)$ . But when  $W > W_c$  or equivalently  $z_2 > z_1 + L(z_1)$ , the charge density goes to zero at the right edge of the support at  $L(z_1) < W$ . A similar type of phase transition was also found in the context of Wishart random matrices [21] and more recently in the context of bipartite entanglement [26].

Having determined the charge density  $\rho_c(\mu) = f(x=\mu + z_1; z_1, W)$ , the saddle-point action  $\Sigma[\rho_c]$ , is then determined via the following equation that is analogous to Eq. (55):

$$\begin{aligned} \Sigma[\rho_c] &= \frac{1}{4}z_1^2 - \frac{1}{2} \int_0^l dx \ln(x) f(x; z_1, W) \\ &+ \frac{1}{4} \int_0^l dx (x+z_1)^2 f(x; z_1, W), \end{aligned} \quad (78)$$

where  $W=z_2-z_1$  and  $f(x; z_1, W)$  and  $l$  are given, respectively, in Eqs. (76) and (77). Denoting the saddle-point action  $S(z_1, W) = \Sigma[\rho_c]$  and evaluating explicitly the integrals in Eq. (78) we get for  $W < L(z_1)$

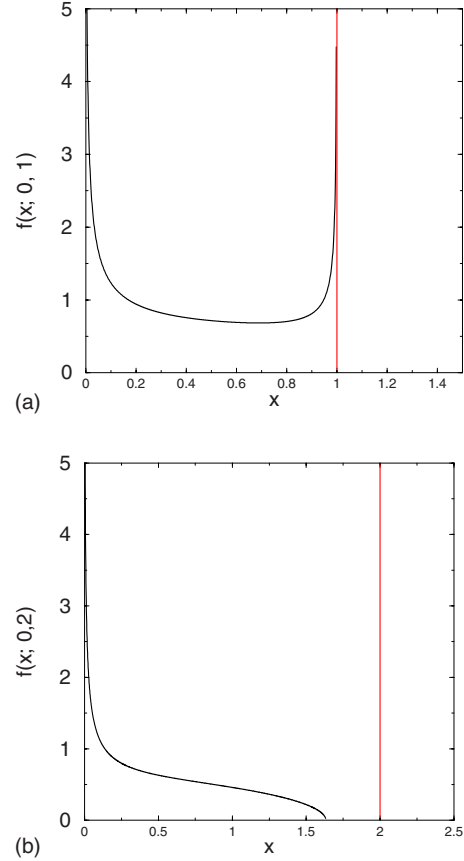


FIG. 3. (Color online): The charge density  $f(x; z_1, W)$  for  $z_1=0$  and  $W=1$  and  $W=2$ , respectively. When the left wall location  $z_1=0$ ,  $L(0) = \sqrt{8/3} = 1.63299$ . Thus the critical value of the distance  $W$  between the walls is  $W_c = 1.63299$ . On the left panel,  $W=1 < W_c$  (subcritical) where the charge density diverges (square root singularity) at the location of second wall  $x=W$ . On the right panel,  $W=2 > W_c$  (supercritical) where the charge density goes to zero as  $x \rightarrow W_c = L(0) < W$ .

$$\begin{aligned} S(z_1, W) &= \frac{1}{32} \left[ 32 \ln(2) - 16 \ln(W) + 16z_1^2 + 6W^2 + 16Wz_1 \right. \\ &\left. - 2W^2z_1^2 - 2W^3z_1 - \frac{9}{16}W^4 \right]. \end{aligned} \quad (79)$$

For  $W > L(z_1)$ ,  $S(z_1, W)$  becomes independent of  $W$  and sticks to its value  $S[z_1, L(z_1)]$ . On the other hand, we know that when  $W \rightarrow \infty$ ,  $S(z_1, \infty)$  must be equal to the action  $S(z_1)$  for a single wall as given in Eq. (56). Indeed, one can check explicitly that  $S[z_1, L(z_1)] = S(z_1)$ , thus confirming the expectation.

The partition function, in terms of the scaled variables  $z_1$  and  $z_2$ , then follows from Eq. (70)

$$\Omega_N(z_1, z_2) = \exp[-\beta N^2 S(z_1, W) + O(N)]. \quad (80)$$

Note that the denominator  $Z_N(-\infty)$  in Eq. (67) is still given by Eq. (57), where  $S(-\sqrt{2}) = [3 + \ln(4)]/8$ . Hence taking the ratio in Eq. (67) and using Eq. (80) we get the joint probability  $R_N(\xi_1, \xi_2)$  for large  $N$

$$R_N(\zeta_1, \zeta_2) = \exp \left[ -\beta N^2 \Psi \left( \frac{\zeta_1}{\sqrt{N}}, \frac{\zeta_2}{\sqrt{N}} \right) + O(N) \right], \quad (81)$$

where

$$\Psi(z_1, z_2) = S(z_1, W = z_2 - z_1) - \frac{3 + \ln(4)}{8} \quad (82)$$

with  $S(z_1, W)$  given by Eq. (79). One can check easily that when the second wall moves to infinity, i.e.,  $z_2 \rightarrow \infty$ ,  $\Psi(z_1, \infty) = \theta(z_1)$ , where  $\theta(z)$  is given in Eq. (59). Thus, Eq. (81) for the joint distribution of  $\lambda_{\min}$  and  $\lambda_{\max}$  is a generalization of Eq. (58) that describes only the distribution of  $\lambda_{\min}$ .

#### IV. NUMERICAL RESULTS

The reader will realize that the numerical confirmation of the analytical results of the previous section is a delicate and potentially computation intensive task. For simplicity we will restrict our selves to the case of the GOE but the methods used can be extended to the other ensembles. The simplest way to compute the probability that all eigenvalues are greater than some value is to numerically generate matrices from the required ensemble, diagonalize them, and then count the number  $m_+$  that satisfy the eigenvalue constraint required. However, because of the order  $N^2$  suppression of this probability found here, for large  $N$  the number of matrices  $m$  that one would need to generate before seeing a single matrix satisfying the constraint is huge. The estimate for the probability that all eigenvalues are positive in this method is given by

$$Q_N(0) = \frac{m_+}{m}. \quad (83)$$

In Ref. [4] an approximate argument for the GOE ( $\beta=1$ ) was made yielding  $\theta=1/4$  and a subsequent numerical study on matrices up to  $7 \times 7$  with an  $N$ -dependent fit  $\theta=aN^\alpha$  yielded  $\alpha=2.00387$  and  $a=0.3291$  [27] was found. However, given that there are  $O(N)$  corrections and the size of the systems studied are so small this fit cannot be taken too seriously. In fact one can use the Coulomb gas representation of the eigenvalues of Gaussian ensembles in order to numerically compute  $\theta(0)$  for much larger values of  $N$ . However, as a test of this method for smaller values of  $N$  we may adopt the direct enumeration approach of [4] but slightly improve it to gain a few extra values of  $N$ .

If all the eigenvalues of a matrix are positive then for any vector  $\mathbf{v}$  we must have that

$$(\mathbf{v}, M\mathbf{v}) > 0. \quad (84)$$

In particular, if we choose the vector  $\mathbf{v}$  to be one of the  $N$  basis vectors  $\mathbf{e}_i$  then Eq. (84) implies that

$$(\mathbf{e}_i, M\mathbf{e}_i) = M_{ii} > 0, \quad (85)$$

and so if  $M$  is positive (in the operator sense), all of the diagonal elements must be positive. The estimation of  $Q_N(0)$  can thus be slightly improved by increasing the chances of seeing a positive matrix by forcing the diagonal elements to be positive. With respect to the simplest form of enumera-

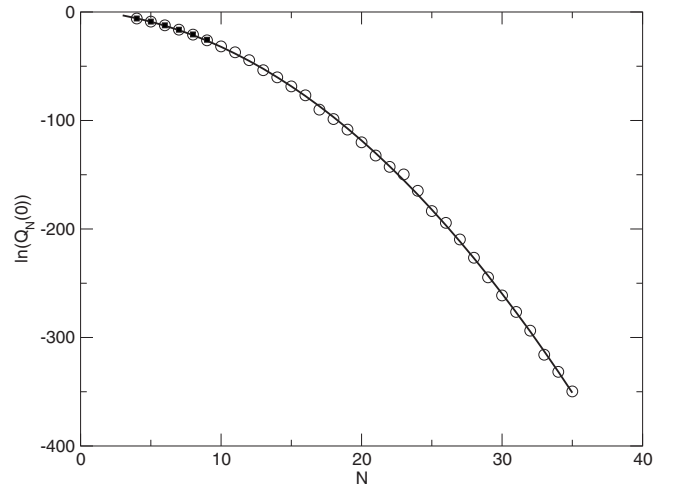


FIG. 4. Monte Carlo computation of  $\ln[Q_N(0)]$  for the GOE (black circles error at large  $N$  indicated by the size of the circles) along with quadratic fit (solid line). Shown as squares are the results obtained by modified enumeration.

tion, the matrices  $M^*$  generated are the same as those for the GOE but the diagonal elements are replaced with their absolute value. The probability that a given matrix  $M$  has all diagonal elements positive is  $1/2^N$ , thus if  $m_+$  denotes the number of these so generated matrices (with positive diagonal elements) then the estimation of the probability that a GOE matrix is positive is given by

$$Q_N(0) = \frac{1}{2^N} \frac{m_+}{m}. \quad (86)$$

For small  $N$  this method thus appreciably increases the probability of generating positive matrices and thus enhances the accuracy of the estimate for  $Q_N(0)$ . Even so using this method it is virtually impossible to obtain meaningful results for  $N > 8$ . The results obtained by this modified enumeration method are shown in Fig. 4 (squares).

For large  $N$  it is in fact much better to evaluate  $Q_N(0)$  directly from Eq. (18) via a Monte Carlo method. We note that we can write

$$\frac{1}{(2\pi)^{N/2}} Z(-\infty) = \langle G(\lambda) \rangle, \quad (87)$$

where  $G(\lambda) = \prod_{i < j} |\lambda_i - \lambda_j|^\beta$  and the angled bracket indicates the average is over  $\lambda_i$  taken to be independent and Gaussian of zero mean and unit variance. The term  $Z(0)$  can also be related to the expectation over  $\lambda_i$  which are similarly independent and Gaussian of zero mean and unit variance but conditioned to be positive, and we denote the average of with respect to these variables by  $\langle \cdot \rangle_+$ . We find that

$$\frac{2^N}{(2\pi)^{N/2}} Z(0) = \langle G(\lambda) \rangle_+, \quad (88)$$

the left-hand side now has the form of a conditional average, the factor of  $2^N$  giving the correct normalization for the conditioned probability distribution. Putting all this together yields

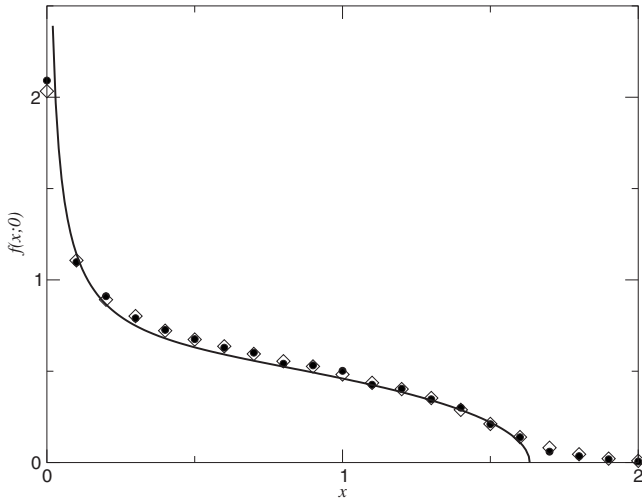


FIG. 5. The analytical large  $N$  formula for  $f(x;0)$  with  $z=0$  (solid line) along with the numerically generated averaged histogram of  $6 \times 6$  (open squares) and  $7 \times 7$  (solid circles) Gaussian matrices with positive eigenvalues. The agreement is already good, the main difference occurring at the large  $x$  tail.

$$Q_N(0) = \frac{1}{2^N} \frac{\langle G(\lambda) \rangle_+}{\langle G(\lambda) \rangle}. \quad (89)$$

The two expectation values can be computed via Monte Carlo sampling, the unconditioned one by using Gaussian random variables and the second trivially by using the absolute value of Gaussian random variables. Shown in Fig. 4 is a plot of  $\ln[Q_N(0)]$  for the GOE ( $\beta=1$ ) ensemble measured as described above (circles), we see that the agreement for small  $N$  with the results obtained by modified enumeration approach is excellent. For larger values of  $N$  we have used  $5 \times 10^8$  Monte Carlo samplings and there is a significant amount of fluctuation as indicated by the error bars. The Monte Carlo results were fitted using the fit  $a x^2 + b x + c$ , for values of  $N$  between 3 and 35, the fit yields  $a = -0.272$ ,  $b = -0.493$ , and  $c = 0.244$  which is in good agreement with that predicted here. However given the errors for large  $N$  and the fact that there are probably corrections of  $O[\ln(N)]$ , the numerical estimate for the exponent probably has about a 10% accuracy.

Also, by direct sampling over Gaussian matrices, one can numerically evaluate the rescaled density for states for matrices having only positive eigenvalues. Because we use the direct sampling method we are clearly restricted to small values of  $N$ , however, in Fig. 5 we show the analytical large  $N$  result for  $f$  with that computed numerically for matrices with  $N=6$  and for  $N=7$ , we see that despite the small value of  $N$  the agreement is already rather good, the main deviation being in the tails for large  $\mu$ .

## V. CONCLUSIONS

In this paper we have shown how the Coulomb gas formulation of the distribution of eigenvalues of  $(N \times N)$  Gaussian random matrices can be exploited to derive exact

asymptotic results concerning the extreme value statistics of their eigenvalues. Our main results are summarized as follows. (i) We have shown the probability  $P_N$  that all eigenvalues are positive (or negative) (or equivalently the probability that  $\lambda_{\min} \geq 0$  or  $\lambda_{\max} \leq 0$ ) decays as  $P_N \sim \exp[-\beta \theta(0) N^2]$  for large  $N$  where  $\theta(0) = \ln(3)/4 = 0.274653\dots$  and  $\beta$  is the Dyson index. (ii) More generally, we have computed the probability  $\text{Prob}[\lambda_{\max} \leq t, N]$  that the maximal eigenvalue is located deep within the Wigner sea region, far to the left from its average value  $\sqrt{2N}$ , i.e., when  $t \sim O(N^{1/2}) \leq \sqrt{2N}$ . This probability has the asymptotic form  $\sim \exp[-\beta N^2 \Phi(\frac{\sqrt{2N}-t}{\sqrt{N}})]$  for large  $N$ , where the large deviation function  $\Phi(y)$  has been computed exactly. (iii) We have also computed the asymptotic joint probability distribution of  $\lambda_{\min}$  and  $\lambda_{\max}$ .

Our result in (ii), valid when  $\sqrt{2N}-t \sim O(N^{1/2})$  (deep inside the Wigner sea) is complimentary to the Tracy-Widom [9] result that concerns the distribution of  $\lambda_{\max}$  about its mean value (near the edge) over a small range of width  $\sim N^{-1/6}$ , i.e., for  $\sqrt{2N}-t \sim O(N^{-1/6})$ . We have demonstrated explicitly how these two results match up smoothly as one approaches from deep inside the Wigner sea to its right edge.

The key step in our method for computing the distribution of  $\lambda_{\min}$  (or  $\lambda_{\max}$ ) in (ii) consists in using a functional integral approach to study the Coulomb gas representation of the problem and imposing a single hard wall constraint which enforces the fact that no eigenvalues can be to the left (or right) of a given point. For the computation of the joint distribution of  $\lambda_{\min}$  and  $\lambda_{\max}$  in (iii), we needed to confine the Coulomb gas within two hard walls. In the limit of large  $N$ , the functional integrals can be evaluated by the saddle-point method and the resulting integral equations for the saddle-point density can be solved explicitly using Tricomi's theorem [24].

Our method is actually rather general and has already been adapted to study the critical points of Gaussian random fields in large-dimensional spaces [19,20] and the extreme value statistics of the maximum eigenvalue of Wishart random matrices [21]. One can possibly find further applications in related statistical problems. For instance, the method is probably adaptable to study the statistics of the index (the number of negative eigenvalues) [6] of random matrices and one could also study the extreme value statistics of the minimal value of the modulus of the eigenvalues by introducing the appropriate constraint on the density of eigenvalues in the functional integral formulation of the problem.

Note that in this paper we were able to compute only the leading large  $N$  behavior of the distribution of extreme eigenvalues. It would be interesting to compute the subleading corrections to this leading behavior. Some recent attempts have been made in this direction [28].

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